SHEET 8 LAPLACE TRANSFORMS

The Laplace Transform of a function *f(t)*, usually time, is denoted by *L{f(t)}* and defined by

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt \equiv \bar{f}(s).$$

The Laplace Transform parameter s is assumed positive and large enough to ensure that the product $f(t)e^{-st}$ converges as $t \to \infty. t \to \infty$.

1. Obtain the Laplace transform of the following functions.

$(a)\ f(t) = a,$ $(b)\ f(t) = e^{at},$ $(c)\ f(t) = \sin at,$

$(d)\ f(t) = \cos at,$ $(e)\ f(t) = \sinh at,$ $(f)\ f(t) = \cosh at,$

$(g)\ f(t) = t,$ $(h)\ f(t) = t^2.$

2. If $L\{f(t)\} = \bar{f}(s)$, show that $L\{e^{-at}f(t)\} = \bar{f}(s+a)$. **This is called the first shift theorem.** Hence write down the Laplace transforms of the following.

$(a)\ L\{e^{-2t}\cosh 3t\},$ $(b)\ L\{e^{2t}\cos t\},$ $(c)\ L\{2e^{3t}\sin 3t\},$

$(d)\ L\{e^{3t}\sinh 2t\},$ $(e)\ L\{4te^{-t}\},$ $(f)\ L\{t^3e^{-4t}\}.$

3. Find the following in tems of $L\{x\}$ (a) $L\left\{\frac{dx}{dt}\right\}$, (b) $L\left\{\frac{d^2x}{dt^2}\right\}$.

4. Use partial fractions to obtain the inverse Laplace transform of the following

(a) $L^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\}$, (b) $L^{-1}\left\{\frac{9s-8}{s^2-2s}\right\}$,

(c) $L^{-1}\left\{\frac{s^2-15s+41}{(s+2)(s-3)^2}\right\}$, (d) $L^{-1}\left\{\frac{4s^2-5s+6}{(s+1)(s^2+4)}\right\}$.

Hint the following forms for partial fractions should be chosen

(a) $\frac{5s+1}{s^2-s-12} \equiv \frac{A}{(s-4)} + \frac{B}{(s+3)},$ (b) $\frac{9s-8}{s^2-2s} \equiv \frac{A}{s} + \frac{B}{s-2}.$

(c) $\left\{\frac{s^2-15s+41}{(s+2)(s-3)^2}\right\} \equiv \frac{A}{s+2} + \frac{B}{s-3} + \frac{c}{(s-3)^2},$

(d) $\left\{\frac{4s^2-5s+6}{(s+1)(s^2+4)}\right\} \equiv \frac{A}{s+1} + \frac{Bs+C}{s^2+4}.$

Answers

[1]. $(a)\ \frac{a}{s}$ $(b)\ \frac{1}{s-a}$ $(c)\ \frac{a}{s^2+a^2}$ $(d)\ \frac{s}{s^2+a^2}$ $(e)\ \frac{a}{s^2-a^2}$ $(f)\ \frac{s}{s^2-a^2}$ $(g)\ \frac{2}{t^3}$

[2]. $(a)\ \frac{s+2}{s^2+4s-5}$ $(b)\ \frac{6}{s^2-6s+18}$ $(c)\ \frac{4}{(s+1)^2}$ $(d)\ \frac{s-2}{s^2-4s+5}$

$(e)\ \frac{2}{s^2-6s+5}$ $(f)\ \frac{6}{(s+4)^4}$