EVEN MORE PARTIAL DIFFERENTIATION
EXERCISE SHEET 8

1. Show that the function $y = \sin\left(\frac{r\pi x}{l}\right)\sin\left(\frac{r\pi ct}{l}\right)$ satisfies the one dimensional wave equation, viz.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}.$$

2. If f is a function of the independent variables x and y and $x = re^{\theta}, y = re^{-\theta}$, show that

$$(a)\ 2x\frac{\partial f}{\partial x} = r\frac{\partial\ f}{\partial r} + \frac{\partial r}{\partial \theta}, \qquad (b)\ 2y\frac{\partial f}{\partial y} = r\frac{\partial f}{\partial r} - \frac{\partial f}{\partial \theta},$$

Deduce that

$$2\left\{x^2\left(\frac{\partial f}{\partial x}\right)^2 + y^2\left(\frac{\partial f}{\partial y}\right)^2\right\} = r^2\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial \theta}\right)^2.$$

3. If $z = f(x, y)$ and $x = e^u \cos v, y = e^u \sin v$ express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ in terms of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ and deduce that

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = (x^2 + y^2)\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}.$$

4. If ϕ is a function of x and y and $u = \ln(x^2 + y^2), v = \frac{y}{x}$, show that

$$(a)\ x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} = 2\frac{\partial \phi}{\partial u},$$

$$(b)\ -y\frac{\partial \phi}{\partial x} + x\frac{\partial \phi}{\partial y} = (1 + v^2)\frac{\partial \phi}{\partial v}.$$